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Quantization of Higher-Order Constrained Lagrangian Systems Using the WKB Approximation

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A general theory is given for solving the Hamilton–Jacobi partial differential equations (HJPDEs) for both constrained and unconstrained systems with arbitrarily higher-order Lagrangians. The Hamilton–Jacobi function is obtained for both types of systems by solving the appropriate set of HJPDEs. This is used to determine the solutions of the equations of motion. The quantization of both systems is then achieved using the WKB approximation. In constrained systems, the constraints become conditions on the wave function to be satisfied in the semiclassical limit.

KEY WORDS: WKB; quantization.

1. INTRODUCTION

Although most physical systems can be described by Lagrangians that depend at most on the first derivatives of the dynamical variables (Sudershan and Mukunda, 1974; Dirac, 1950, 1964), there is a continuing interest in the so-called generalized dynamics; that is, the study of physical systems described by Lagrangians containing derivatives of order higher than one.

Theories associated with higher-order regular Lagrangians were first developed by Ostrogradski (1850). These led to Euler's and Hamilton's equations of motion.

A new formalism for investigating first-order singular systems—the canonical—was developed by Rabei and Guler (1992). These authors obtained a set of Hamilton–Jacobi partial differential equations for such systems using Caratheodory's (1967) equivalent Lagrangian method. Recently the formalism was extended to second- and higher-order Lagrangians (Pimentel and Teixeira, 1996, 1998).

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The quantization of constrained systems has been studied for *first-order* singular Lagrangians using the WKB approximation (Rabei *et al.*, 2002). The HJPDEs for these systems have been constructed using the canonical method. The Hamilton–Jacobi functions have then been obtained by solving these equations.

The aim of this paper is to study the quantization of singular systems with *arbitrarily higher-order* Lagrangians using the WKB approximation.

The paper is organized as follows. In Section 2, the Hamilton–Jacobi formulation is reviewed briefly for both constrained and unconstrained systems with higher-order Lagrangians. A generalized method is proposed for determining the Hamilton–Jacobi function for both types of systems. The equations of motion are then derived from this function. In Section 3, the WKB approximation for both types of systems is introduced. The work closes with some concluding remarks in Section 5.

2. HAMILTON–JACOBI FORMULATION FOR HIGHER-ORDER LAGRANGIANS

The starting point is a system described by a Lagrangian dependent on up to the *Kth* derivative of N generalized coordinates q_i; i.e.,

$$L \equiv L\big(q_i, \dot{q}_i, \ldots, \overset{(K)}{q_i}\big); \quad q_{(s)i} = \overset{(s)}{q_i} = \frac{d^s q_i}{dt^s}, \tag{2.1}$$

where $s = 0, 1, \ldots, K$ and $i = 1, \ldots, N$. For such systems the Euler–Lagrange equations of motion are obtained through Hamilton's principle of stationary action:

$$\sum_{s=0}^{K} (-1)^s \frac{d^s}{dt^s}\left(\frac{\partial L}{\partial \overset{(s)}{q}_i}\right) = 0. \tag{2.2}$$

This is a system of N differential equations of $2Kth$ order; so we need $2KN$ initial conditions to solve it. These conditions are the initial values of $q_i, \dot{q}_i, \ldots, \overset{(2K-1)}{q_i}$ that describe the velocity phase space (VPS).

The Hamiltonian formalism for theories with higher-order derivatives, first developed by Ostrogradski (1850), treats the derivatives $\overset{(s)}{q_i}(s = 0, \ldots, K-1)$ as coordinates. We will indicate this by writing them as $\overset{(s)}{q_i} \equiv q_{(s)i}$. In Ostrogradski's formalism the momenta conjugated, respectively, to $\overset{(K-1)}{q_i}$ and $\overset{(s-1)}{q_i}(s = 1, \ldots, K-1)$ are introduced as

$$p_{(K-1)i} \equiv \frac{\partial L}{\partial \overset{(K)}{q_i}}; \tag{2.3a}$$

$$p_{(s-1)i} \equiv \frac{\partial L}{\partial \overset{(s)}{q_i}} - \dot{p}_{(s)i}, \quad (s = 1, \ldots, K-1). \tag{2.3b}$$

Clearly, the momenta $p_{(s)i}$ ($s \geq 0$) will only be dependent on the derivatives up to $\overset{(2K-1-s)}{q_i}$.

The Hamiltonian is defined as

$$H = \sum_{s=0}^{K-1} p_{(s)i} \overset{(s+1)}{q_i} - L\left(q_i, \ldots, \overset{(K)}{q_i}\right); \tag{2.4}$$

here and throughout the paper we use Einstein's summation rule for repeated indices.

Hamilton's equations of motion can be written as

$$\dot{q}_{(s)i} = \frac{\partial H}{\partial p_{(s)i}} = \{q_{(s)i}, H\}; \tag{2.5}$$

$$\dot{p}_{(s)i} = -\frac{\partial H}{\partial q_{(s)i}} = \{p_{(s)i}, H\}. \tag{2.6}$$

where {,} is the Poisson bracket defined as

$$\{A, B\} \equiv \sum_{s=0}^{K-1} \frac{\partial A}{\partial q_{(s)i}} \frac{\partial B}{\partial p_{(s)i}} - \frac{\partial B}{\partial q_{(s)i}} \frac{\partial A}{\partial p_{(s)i}}. \tag{2.7}$$

The fundemental Poisson brackets are

$$\left\{q_{(s)i}, p_{(s')j}\right\} = \delta_{ss'}\delta_{ij}; \quad \left\{q_{(s)i}, q_{(s')j}\right\} = \left\{p_{(s)i}, p_{(s')j}\right\} = 0, \tag{2.8}$$

where $i, j = 1, \ldots, N$ and $s, s' = 0, \ldots, K-1$. With this procedure the phase space (PS) is described in terms of the canonical variables $q_{(s)i}$ and $p_{(s)i}$ ($i = 1, \ldots, N$ and $s = 0, \ldots, K-1$) obeying $2KN$ equations of motion given by Equations (2.5) and (2.6), which are first-order differential equations.

However, this transformation from (VPS) to (PS) is possible only if we can solve the momenta expression, Equation (2.3), with recpect to the derivatives $\overset{(K)}{q_i}, \ldots, \overset{(2K-1)}{q_i}$; so that these can be expressed as functions of the canonical variables and eliminated from the theory.

Prior to this, we will give a brief review of Caratheodory's (1967) equivalent Lagrangian method to extend the Hamilton–Jacobi formalism to a general higher-order Lagrangian (Pimentel and Teixeira, 1998). This formalism can be applied to any higher-order Lagrangian and is not limited to singular cases.

Let us consider a Lagrangian $L(q_i, \dot{q}_i, \ldots, \overset{(K)}{q_i}, t)$. One can obtain a completely equivalent Lagrangian by introducing

$$L' = L\left(q_i, \ldots, \overset{(K)}{q}_i, t\right) - \frac{dS(q_i, \ldots, q_{(K-1)i}, t)}{dt}, \tag{2.9}$$

such that the auxiliary function $S(q_i, \ldots, q_{(K-1)i}, t)$ must satisfy

$$\frac{\partial S}{\partial t} = -H_0, \tag{2.10}$$

where H_0 is defined as the usual Hamiltonian:

$$H_0 = \sum_{s=0}^{K-1} p_{(s)i} \overset{(s+1)}{q_i} - L\left(q_i, \ldots, \overset{(K)}{q_i}\right); \tag{2.11}$$

and the momenta $p_{(u)i}$ are given by

$$p_{(s)i} = \frac{\partial S}{\partial q_{(s)i}}, \quad s = 0, \ldots, K-1. \tag{2.12}$$

These are the fundamental equations of the equivalent Lagrangian method; Equation (2.10) is the relevant Hamilton–Jacobi partial differential equation.

Now the extended Hessian matrix is defined as

$$A_{ij} = \frac{\partial^2 L}{\partial \overset{(K)}{q_i}\, \partial \overset{(K)}{q_j}}. \tag{2.13}$$

For a regular system, the Hessian matrix has rank N and the canonical coordinates are independent. For the singular Lagrangian case, the Hessian matrix has rank N−R, R<N. In this case R of the momenta are dependent. The generalized momenta conjugate to the generalized coordinates $q_{(K-1)i}$ are defined as

$$p_{(K-1)a} = \frac{\partial L}{\partial \overset{(K)}{q_a}}, \quad a = R+1, \ldots, N; \tag{2.14}$$

$$p_{(K-1)\mu} = \frac{\partial L}{\partial \overset{(K)}{q_\mu}}, \quad \mu = 1, \ldots, R. \tag{2.15}$$

Since the rank of the Hessian matrix is N−R, one can solve Equation (2.14) for $q_{(K-1)\mu}$ as functions of $q_{(s)i}$, $p_{(K-1)a}$ and $\overset{(K)}{q_\mu}$ as follows:

$$\overset{(K)}{q_a} = w_{(K)a}\left(q_{(s)i}, p_{(K-1)a}, \overset{(K)}{q_\mu}\right). \tag{2.16}$$

Substituting Equation (2.16) into (2.15), one gets

$$p_{(K-1)\mu} = \left.\frac{\partial L}{\partial \overset{(K)}{q_\mu}}\right|_{\overset{(K)}{q_a} = w_{(K)a}\left(q_{(s)i}, p_{(K-1)a}, \overset{(K)}{q_\mu}\right)}; \tag{2.17a}$$

or

$$p_{(s)\mu} = -H_{(s)\mu}\left(t, q_{(u)j}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}\right); \quad u, s = 0, \ldots, K-1,$$

$$j = 1, \ldots, N, \tag{2.17b}$$

which are called primary constraints (Dirac, 1964; Ostrogradski, 1850).

The coordinates t are replaced as $t_{(s)0} \equiv q_{(s)0}$ (for any value of s); and the coordinates $t_{(s)\mu}$ will be called $q_{(s)\mu}$. Further, we write $p_{(s)0} \equiv \frac{\partial S}{\partial t}$, and $H_{(s)0} \equiv H_0$ for any value of s. The canonical Hamiltonian H_0 can then be written as

$$\begin{aligned} H_0 = &\sum_{u=0}^{K-2} p_{(u)a} \overset{(u+1)}{q_a} + p_{(K-1)a} w_{(K)a} + \sum_{u=0}^{K-1} \overset{(u+1)}{q_\mu} \\ &\times p_{(u)\mu}|_{p_{(s)\upsilon}=H_{(s)\upsilon}} - L\big(q_{(s)i}, \ldots, \overset{(K)}{q_\mu}, \overset{(K)}{q_a} = w_{(K)a}\big); \\ &\mu, \upsilon = 1, \ldots, R; \quad a = R+1, \ldots, N. \end{aligned} \tag{2.18}$$

The canonical method leads to the following set of HJPDEs:

$$\begin{aligned} H_0' = H_{(s)0}' = P_{(s)0} + H_{(s)0} = P_{(s)0} \\ + H_{(s)0}\left(t, q_{(u)\mu}; q_{(u)a}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}\right) = 0; \end{aligned} \tag{2.19a}$$

$$\begin{aligned} H_{(s)\mu}' = p_{(s)\mu} + H_{(s)\mu} = p_{(s)\mu} + H_{(s)\mu}\left(q_{(u)\mu}; q_{(u)a}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}\right) = 0; \\ \times u, s = 0, \ldots, K-1; \quad \mu = 1, \ldots, R; \end{aligned} \tag{2.19b}$$

or

$$\begin{aligned} H_{(s)\alpha}' = p_{(s)\alpha} + H_{(s)\alpha} = p_{(s)\alpha} + H_{(s)\alpha}\left(q_{(u)\beta}; q_{(u)a}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}\right) = 0; \\ \alpha, \beta = 0, 1, \ldots, R. \end{aligned} \tag{2.19c}$$

The equations of motion are written as total differential equations in many variables as follows:

$$dq_{(u)i} = \sum_{s=0}^{K-1} \frac{\partial H_{(s)\alpha}'}{\partial p_{(u)i}}\, dt_{(s)\alpha}; \tag{2.20a}$$

$$dp_{(u)i} = -\sum_{s=0}^{K-1} \frac{\partial H_{(s)\alpha}'}{\partial q_{(u)i}}\, dt_{(s)\alpha}; \tag{2.20b}$$

The set of Equations (2.20) is integrable (Rabei and Guler, 1992; Pimentel and Teixeira, 1998) if and only if

$$dH_{(s)\alpha}' \equiv 0, \quad \alpha = 0, 1, \ldots, R, \quad s = 0, \ldots, K-1; \tag{2.21}$$

or it leads to new secondary constraints (Dirac, 1950, 1964). In the case of new constraints, one should consider their total variations also. Repeating this procedure,

one may then obtain a set of constraints such that all the total variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the solutions of a singular system.

2.1. Determination of the Hamilton–Jacobi Function for Higher-Order Lagrangians

2.1.1. Unconstrained Systems

Under certain conditions it is possible to separate the variables in the Hamilton–Jacobi equations, and the solution can then always be reduced to quadratures (Goldstein, 1980; Arnold, 1989; Brack and Bhaduri, 1997). In practice, the Hamilton–Jacobi technique becomes a useful computational tool only when such a separation can be effected. In general, coordinates $q_{(s)i}$ are said to be separable in the Hamilton–Jacobi equations when Hamilton's principal function can be split into two additive parts: one that depends only on the generalized coordinates $q_{(s)i}$, and another that is entirely independent of these derivatives.

In the cases to which we shall apply the method of separation of variables, the Hamiltonian will be time-independent. If we then restrict our considerations to such Hamiltonians, the Hamilton–Jacobi equation for higher-order unconstrained systems will be

$$\frac{\partial S\left(q_{(s)i}, t\right)}{\partial t} + H_0\left(q_{(s)i}, p_{(s)i} = \frac{\partial S}{\partial q_{(s)i}}\right) = 0, \quad i = 1, \ldots, N, \quad s = 0, \ldots, K-1. \tag{2.22}$$

We shall first try to find a solution that can be written in separable form:

$$S\left(q_{(s)i}, t\right) = \sum_{s=0}^{K-1} W_{(s)i}\left(q_{(s)i}\right) + f(t). \tag{2.23}$$

Substituting this into Equation (2.22), we get

$$\frac{df}{dt} = -H_0\left(q_{(s)i}, p_{(s)i} = \frac{\partial S}{\partial q_{(s)i}}\right). \tag{2.24}$$

The left-hand side depends only on t; whereas the right-hand side depends only on the *Kth* derivatives of the generalized coordinates $q_{(s)i}$; therefore, each side must be equal to a constant independent of $q_{(s)i}$ and t. Let this constant be $-E_{(K-1)}$. We then have

$$f(t) = -E_{(K-1)}t = -\sum_{i=1}^{N} E_{(K-1)i}\,t,$$

where $E_{(K-1)} = \sum_{i=1}^{N} E_{(K-1)i}$. We can then write the Hamilton–Jacobi function

$$S(q_{(s)i}, t) = \sum_{s=0}^{K-1} W_{(s)i}\left(q_{(s)i}\right) - E_{(K-1)}t, \tag{2.25}$$

and the following equation for $W_{(s)}$ (s is just the numbers: $s = 0, \ldots, K-1$):

$$H_0\left(q_{(s)i}, p_{(s)i} = \frac{\partial S}{\partial q_{(s)i}}\right) = E_{(K-1)}, \tag{2.26}$$

This shows that, for time-independent Hamiltonians, we can always separate out the time. We can proceed further using the method of separation of variables only if Equation (2.26) is similarly separable in each of the $q_{(s)i}$; that is, if a solution can be written in the form

$$W = \sum_i \left[\sum_{s=0}^{K-1} W_{(s)i}\left(q_{(s)i}, E_{(s)i}\right)\right], \quad i = 1, \ldots, N. \tag{2.27}$$

Once we have found the Hamilton–Jacobi function S, the equations of motion can be obtained by using the so-called canonical transformations (Goldstein, 1980; Arnold, 1989), as follows:

$$\lambda_{(s)i} = \frac{\partial S}{\partial E_{(s)i}}; \tag{2.28a}$$

$$p_{(s)i} = \frac{\partial S}{\partial q_{(s)i}}, \quad s = 0, \ldots, K-1, \tag{2.28b}$$

where $\lambda_{(s)i}$ are constants and can be determined from the initial conditions.

One can finally solve Equations (2.28) to get

$$q_{(s)i} = q_{(s)i}\left(\lambda_{(s)i}, E_{(s)i}, t\right); \tag{2.29a}$$

$$p_{(s)i} = p_{(s)i}\left(\lambda_{(s)i}, E_{(s)i}, t\right). \tag{2.29b}$$

2.1.2. Constrained Systems

In this case, instead of considering the Hamilton–Jacobi equation (2.10), we shall be dealing with a set of HJPDEs; Equations (2.19c). If we have the same conditions for separable coordinates and follow the same procedure just discussed, we can extend this method to constrained systems. Moreover, because of the singular nature of the dynamical Lagrangians, we should split the $q_{(s)i}$ coordinates of the system into those corresponding to independent momenta, $q_{(s)a}$, and others corresponding to dependent momenta, $q_{(s)\mu}$. Thus, we can guess a

general solution for Equations (2.22) in the form

$$S\big(q_{(s)a}, q_{(s)\mu}, t\big) = f(t) + \sum_{s=0}^{K-1} \big[W_{(s)a}\big(q_{(s)a}, E_{(s)a}\big) + f_{(s)\mu}\big(q_{(s)\mu}\big)\big] + A, \quad (2.30)$$

where $f(t) = -E_{(K-1)}t = -\sum_{a=1}^{N-R} E_{(K-1)a}t$. Here $q_{(s)\mu}$ are treated as independent variables, just as the time t.

Once we have found the Hamilton–Jacobi function S, the equations of motion can be obtained in the manner of regular systems, using the so-called canonical transformations (Goldstein, 1980; Arnold, 1989), as follows:

$$\lambda_{(s)a} = \frac{\partial S}{\partial E_{(s)a}}; \quad (2.31a)$$

$$p_{(s)i} = \frac{\partial S}{\partial q_{(s)i}}; \quad s = 0, \ldots, K-1, \quad (2.31b)$$

where $\lambda_{(s)a}$ are constants and can be determined from the initial conditions.

One can solve Equations (2.31) to get

$$q_{(s)a} = q_{(s)a}\big(\lambda_{(s)a}, E_{(s)a}, q_{(s)\mu}, t\big); \quad (2.32b)$$

$$p_{(s)i} = p_{(s)i}\big(\lambda_{(s)a}, E_{(s)a}, q_{(s)\mu}, t\big). \quad (2.32c)$$

From the initial conditions, one can then determine the constants $\lambda_{(s)a}$.

To this end, further insight into the physical significance of $S(q_i, \ldots, q_{(K-1)i}, t)$ is gained by an examination of its total time derivative:

$$\frac{dS}{dt} = \sum_{s=0}^{(K-1)} \left(\frac{\partial S}{\partial q_{(s)a}} q_{(s+1)a} + \frac{\partial S}{\partial q_{(s)\mu}} q_{(s+1)\mu} \right) + \frac{\partial S}{\partial t} \quad (2.33a)$$

$$\frac{dS}{dt} = \sum_{s=0}^{(K-1)} \big(p_{(s)a} q_{(s+1)a} + p_{(s)\mu} q_{(s+1)\mu}\big) - H_0 = L. \quad (2.33b)$$

Thus, Hamilton's principal function differs from the time integral of the Lagrangian only by a constant:

$$S = \int L dt + \text{constant}. \quad (2.34)$$

In actual caculations, however, one cannot find S in terms of time directly from this integral unless $q_{(s)i}$ and $p_{(s)i}$ are known as functions of time.

3. QUANTIZATION USING THE WKB APPROXIMATION

It is well known that the H–J equation for first-order unconstrained Lagrangian systems leads naturally to a semiclassical approximation, namely, WKB, that is

very successful in integrable problems. Recently it has been shown (Rabei *et al.*, 2002) that this approximation is equally applicable to first-order constrained Lagrangian systems. We shall see that this is also valid for both types of systems with arbitrarily higher-order Lagrangians.

3.1. Unconstrained Systems

First we shall show how quantum mechanics reproduces the Hamilton–Jacobi equations of classical mechanics in the semiclassical limit $\hbar \to 0$.

The Schrödinger equation for a single particle in a potential $V(q_{(s)})$ is

$$i\hbar\frac{\partial\Psi(q_{(s)},t)}{\partial t}=\left[\sum_{s=0}^{K-1}\left[-\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial q_{(s)}^2}\right)+V(q_{(s)})\right]\right]\Psi(q_{(s)},t),$$
$$s=0,\ldots,K-1. \tag{3.1}$$

Using the substitution (Griffiths, 1995)

$$\Psi(q_{(s)},t)=\exp\left(\frac{iS\big(q_{(s)},t\big)}{\hbar}\right), \tag{3.2}$$

one can write (3.1) as

$$-\frac{\partial S}{\partial t}\Psi=\left[\sum_{s=0}^{K-1}\left[\frac{1}{2}\left(\frac{\partial S}{\partial q_{(s)}}\right)^2-\frac{i\hbar}{2}\frac{\partial^2 S}{\partial q_{(s)}^2}+V\big(q_{(s)}\big)\right]\right]\Psi. \tag{3.3}$$

Assuming $\Psi\neq 0$, this leads to

$$-\frac{\partial S}{\partial t}=\sum_{s=0}^{K-1}\left[\frac{1}{2}\left(\frac{\partial S}{\partial q_{(s)}}\right)^2-\frac{i\hbar}{2}\frac{\partial^2 S}{\partial q_{(s)}^2}+V\big(q_{(s)}\big)\right]. \tag{3.4}$$

Taking the formal limit $\hbar \to 0$, we obtain the classical Hamilton–Jacobi equation

$$-\frac{\partial S}{\partial t}=\sum_{s=0}^{K-1}\left[\frac{1}{2}\left(\frac{\partial S}{\partial q_{(s)}}\right)^2+V\big(q_{(s)}\big)\right]. \tag{3.5}$$

Thus, in this limit, quantum mechanics reduces to classical mechanics.

Now that we have rederived classical mechanics from quantum mechanics, our interest turns to quantum mechanics itself .

Equation (3.3) is exact so long as $\Psi\neq 0$. One can use this equation and consider the expansion

$$S\big(q_{(s)},t\big)=S_0+\hbar S_1+\hbar^2 S_2+\cdots. \tag{3.6}$$

This is the so-called $\hbar$, or semiclassical, expansion. Plugging it into Equation (3.4), we find

$$-\frac{\partial S_0}{\partial t} = \sum_{s=0}^{K-1}\left[\frac{1}{2}\left(\frac{\partial S_0}{\partial q_{(s)}}\right)^2 + V\big(q_{(s)}\big)\right]; \tag{3.7}$$

$$-\frac{\partial S_1}{\partial t} = \frac{1}{2}\sum_{s=0}^{K-1}\left[-i\frac{\partial^2 S_0}{\partial q_{(s)}^2} + 2\left(\frac{\partial S_0}{\partial q_{(s)}}\right)\left(\frac{\partial S_1}{\partial q_{(s)}}\right)\right]. \tag{3.8}$$

and similarly for higher terms in $\hbar$. The leading equation has only S_0, and it is exactly the same as the Hamilton–Jacobi equation.

The WKB approximation is used mostly for time-independent cases; in other words, for an eigenstate of energy E. The wave function has then the ordinary time-dependence:

$$\exp\left(\frac{-iEt}{\hbar}\right). \tag{3.9}$$

For K-order Lagrangians, the Hamilton–Jacobi function S takes the form

$$S\big(q_{(s)}, t\big) = \sum_{s=0}^{K-1} S_{(v)}^{(s)}\big(q_{(s)}\big) - Et. \tag{3.10}$$

Therefore, only S_0 has time-dependence ($v = 0, 1$):

$$S_0\big(q_{(s)}, t\big) = \sum_{s=0}^{K-1} S_0^{(s)}\big(q_{(s)}\big) - \sum_{s=0}^{K-1} E_{(s)}t;; \tag{3.11a}$$

so that

$$S_1\big(q_{(s)}\big) = \sum_{s=0}^{K-1} S_1^{(s)}\big(q_{(s)}\big), \quad s = 0, \ldots, K-1; \tag{3.11b}$$

while higher-order terms do not depend on time. The lowest term S_0 in Equations (3.7) satisfies the Hamilton–Jacobi equations

$$E_{(s)} = \frac{1}{2}\left(\frac{\partial S_0^{(s)}}{\partial q_{(s)}}\right)^2 + V\big(q_{(s)}\big). \tag{3.12}$$

The solutions of these differential equations are

$$S_0^{(s)}\big(q_{(s)}\big) = \pm\int\sqrt{2\big(E_{(s)} - V\big(q_{(s)}\big)\big)}dq'_{(s)} = \int p_{(s)}\big(q'_{(s)}\big)\,dq'_{(s)}. \tag{3.13}$$

We have used the notation $p_{(s)}(q_{(s)})$ because these are nothing but the linear momenta of the particles in the classical sense. Once we have known $S_0^{(s)}$, we can

solve for $S_1^{(s)}$. Starting from Equation (3.8) and using

$$\frac{\partial S_1}{\partial t} = 0,$$

we find

$$2\left(\frac{\partial S_0^{(s)}}{\partial q_{(s)}}\right)\left(\frac{\partial S_1^{(s)}}{\partial q_{(s)}}\right) - i\left(\frac{\partial^2 S_0^{(s)}}{\partial q_{(s)}^2}\right) = 0. \tag{3.14}$$

These have the solutions

$$S_1^{(s)}\big(q_{(s)}\big) = \frac{i}{2}\ln p_{(s)}\big(q_{(s)}v\big) + \text{constant}. \tag{3.15}$$

Therefore, the general solution of Schrödinger's equation up to this order is

$$\Psi\big(q_{(s)}, t\big) = \exp\left[\sum_{s=0}^{K-1}\left[\frac{i}{\hbar}\big(S_0^{(s)}\big) + i\big(S_1^{(s)}\big)\right]\right]\exp\frac{-iEt}{\hbar} \quad \big(E = E_{(s)}\big) \tag{3.16}$$

$$\begin{aligned}\Psi\big(q_{(s)}, t\big) \;&=\; \prod_{s=0}^{(K-1)} \frac{A'}{\sqrt{p_{(s)}\big(q_{(s)}\big)}}\exp\sum_{s=0}^{(K-1)} \\ &\times\left[\pm\frac{i}{\hbar}\int\sqrt{2\big(E_{(s)} - V\big(q_{(s)}\big)\big)}dq_{(s)}\right]\exp\left(\frac{-iEt}{\hbar}\right).\end{aligned} \tag{3.17}$$

The overall constant A' is, of course, undetermined from the foregoing analysis. This solution makes it immediately clear that the present approximation breaks down when $p_{(s)}(q_{(s)})$ goes to zero. Here we take the first two terms in the semiclassical expansion because the higher terms do not affect the amplitude $A(\vec{q}, \dot{\vec{q}}, t)$.

For K-order Lagrangians, the transformation to the N-dimensional case is achieved by expanding the wave function Ψ as

$$\Psi(q_{(s)i}, t)\,\Psi(q_{(s)i}, t) = \left[\prod_{i=1}^{N}\prod_{s=0}^{K-1}\psi_{0i}^{(s)}\big(q_{(s)i}\big)\right]\exp\left(\frac{iS\big(q_{(s)i}, t\big)}{\hbar}\right),$$
$$s = 0, \ldots, K-1, \tag{3.18}$$

where

$$\psi_{0i}^{(s)}\big(q_{(s)i}\big) = \frac{1}{\sqrt{p_{(s)i}\big(q_{(s)i}\big)}}.$$

This wave function represents our main result. It satisfies the following condition:

$$\hat{H}_0'\Psi = 0. \tag{3.19}$$

This is obtained when the dynamical coordinates and momenta are turned into their corresponding operators:

$$q_{(s)i} \to q_{(s)i};$$
$$p_{(s)i} \to \hat{p}_{(s)i} = \frac{\hbar}{i}\frac{\partial}{\partial q_{(s)i}};$$
$$p_{(s)0} \to \hat{p}_{(s)0} = \frac{\hbar}{i}\frac{\partial}{\partial t}.$$

3.2. Constrained Systems

For constrained systems with higher-order Lagrangians; the rank of the Hessian matrix is N − R. Thus, the wave function reduces to

$$\Psi = \left[\prod_{a=1}^{N-R}\prod_{s=0}^{K-1}\psi_{0a}^{(s)}\left(q_{(s)a}\right)\right]\exp\left(\frac{iS\left(q_{(s)a}, q_{(s)\mu}, t\right)}{\hbar}\right), \tag{3.20}$$

where

$$\psi_{oa}^{(s)}\left(q_{(s)a}\right) = \frac{1}{\sqrt{p_{(s)a}\left(q_{(s)a}\right)}}, \quad s = 0, \ldots, K-1.$$

This wave function represents our main result. It satisfies the following conditions:

$$\widehat{H}'_{(s)\alpha}\Psi = 0; \quad \alpha = 0, 1, \ldots, R, \quad s = 0, \ldots, K-1. \tag{3.21}$$

These are obtained when the dynamical coordinates and momenta are turned into their corresponding operators:

$$q_{(s)i} \to q_{(s)i};$$
$$p_{(s)i} \to \hat{p}_{(s)i} = \frac{\hbar}{i}\frac{\partial}{\partial q_{(s)i}};$$
$$p_{(s)0} \to \hat{p}_{(s)0} \to \hat{p}_0 = \frac{\hbar}{i}\frac{\partial}{\partial t}. \quad s = 0, \ldots, K-1.$$

In passing, it is interesting to use the representation

$$\Psi = A\left(q_{(s)}, t\right)\exp\left(\frac{iS\left(q_{(s)}, t\right)}{\hbar}\right), \quad s = 0, \ldots, K-1.$$

which is simply the so-called Madelung transformation (Ghassib, 1986). Substituting back into the Schrödinger Equation (3.1), one can split up the resulting equation into two real equations by separating the real and imaginary parts.

The real part leads to the equation

$$\frac{\partial S}{\partial t} + \sum_{s=0}^{K-1} \left[\frac{1}{2m} \left(\vec{\nabla} S \right)^2_{q_{(s)}} + V \left(\underset{(s)}{\vec{q}} \right) - \frac{\hbar^2}{2m} \frac{\nabla^2_{q_{(s)}} A}{A} \right] = 0, \quad s = 0, \ldots, K-1.$$

This is the quantum Hamilton–Jacobi equation. In addition to the kinetic energy and the classical potential V, the Hamilton contains a new term, the well-known quantum potential Q:

$$Q(\underset{(s)}{\vec{q}}, t) \equiv \sum_{s=0}^{K-1} \left[-\frac{\hbar^2}{2m} \frac{\nabla^2_{q_{(s)}} A}{A} \right], \quad s = 0, \ldots, K-1.$$

For constrained systems, the quantum potential can be treated in the same manner as for regular systems. Clearly, setting $Q = 0$, one gets back the classical Hamilton–Jacobi equation. This means that the classical limit can be defined as the case in which the quantum potential may be suppressed.

On the other hand, the imaginary part gives the continuity equation

$$\frac{\partial A^2}{\partial t} + \sum_{s=0}^{K-1} \vec{\nabla} q_{(S)} \cdot \left(A^2 \frac{\vec{\nabla} q_{(s)} S}{m} \right) = 0, \quad s = 0, \ldots, K-1.$$

Here $A^2(q_{(s)}, t)$ is the probability density, and the expression inside the parentheses represents the standard definition of the current density.

4. CONCLUSION

This work has aimed at, first, determining the Hamilton–Jacobi function S for both constrained and unconstrained systems with arbitrarily higher-order Lagrangians; and, second, quantizing these systems using the WKB approximation.

We have obtained the quantization of singular systems with arbitrarily higher-order Lagrangians using the WKB approximation starting from the Hamilton–Jacobi function S in configuration space under the conditions that the set of HJPDEs is integrable. The equations of motion are furnished out using the function S. These solutions are obtained in terms of the time and the spatial coordinates that correspond to dependent momenta; these are treated as independent variables, just as the time t.

This is followed by determining the suitable wave function for both types of systems. In constrained systems, the constraints become conditions on the wave function to be satisfied in the semiclassical limit, in addition to the Schrödinger equation.

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